Fourier Analysis 03-16

Review

Thm 2 (Weyl's criterion). Let $(x_n)_{n=1}^{\infty} \subset [0,1)$.
Then (x_n) is equidistributed in $[0,1)$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0 \text{ for all } k \in \mathbb{Z}\setminus\{0\}.$$

Thm 3. Let $0<\alpha<1$. Then

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i 2^n x}, \quad x \in \mathbb{R}$$

is cts but nowhere differentiable.

Prop 4. For any $g \in \mathcal{R}[-\pi,\pi]$, if g is diff at x_0,

then

$$\Delta_N(g)'(x_0) = O(\log N),$$

where $\Delta_N(g) = 2\sigma_{2N}(g) - \sigma_N(g)$.

As we proved in last class, Prop 4 $\Rightarrow$ Thm 3.

Lemma 5: Let $F_N(x) = \sum_{|n|\le N} \left(1-\frac{|n|}{N}\right) e^{inx}$

$$= \frac{\sin^2 \frac{Nx}{2}}{N \sin^2 \frac{x}{2}}.$$

Then $\exists$ a constant $A > 0$ such that

$$|F_N'(x)| \le AN^2, \quad |F_N'(x)| \le \frac{A}{x^2}$$

for any $x \in [-\pi, \pi]$

Proof of Prop 4.

Since $\Delta_N(g) = 2\sigma_{2N}(g) - \sigma_N(g)$,

it suffices to show that

$$\sigma_N(g)'(x_0) = O(\log N).$$

Recall that

$$\sigma_N(g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x-t)\cdot g(t)\,dt$$

Hence

$$\sigma_N(g)'(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N'(x-t)\, g(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N'(t)\, g(x-t)\, dt$$

In particular,

$$\sigma_N(g)'(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N'(t)\, g(x_0-t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N'(t)\big(g(x_0-t) - g(x_0)\big)\, dt$$

(because $\int_{-\pi}^{\pi} F_N'(t)\, dt = F_N(t)\Big|_{-\pi}^{\pi} = 0$)

Hence

$$\left|\sigma_N(g)'(x_0)\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F_N'(t)\right| \cdot \left|g(x_0-t) - g(x_0)\right| dt$$

Since g is diff at x_0, $g \in \mathcal{R}[-\pi, \pi]$, we see that

$$\left|\frac{g(x_0-t) - g(x_0)}{t}\right| \leq \text{const} \quad \text{on } t \in [-\pi, \pi]$$

We obtain that

$$\left|\sigma_N(g)'(x_0)\right| \leq C \int_{-\pi}^{\pi} |F_N'(t)|\ |t|\ dt$$

Notice that

$$\int_{-\pi}^{\pi} |F_N'(t)||t|\, dt = \int_{|t|\leq\frac{1}{N}} + \int_{\frac{1}{N}<|t|\leq\pi} |F_N'(t)|\,|t|\,dt$$

$$= (\mathrm{I}) + (\mathrm{II})$$

Now

$$(\mathrm{I}) = \int_{|t|\leq\frac{1}{N}} |F_N'(t)|\ |t|\,dt$$

$$\leq \int_{|t|\leq\frac{1}{N}} AN^2\cdot\frac{1}{N}\,dt = 2A.$$

$$(\mathrm{II}) = \int_{\frac{1}{N}<|t|\leq\pi} |F_N'(t)|\cdot|t|\,dt$$

$$\leq \int_{\frac{1}{N}<|t|\leq\pi} \frac{A}{t^2}\cdot|t|\,dt = 2\int_{\frac{1}{N}}^{\pi}\frac{A}{t}\,dt$$

$$= 2A\log t\Big|_{\frac{1}{N}}^{\pi}$$

$$= 2A\left(\log \pi + \log N\right)$$

Hence

$$(\mathrm{I}) + (\mathrm{II}) \leq 2A\left(1 + \log \pi + \log N\right)$$
$$= O(\log N).$$

Thm 3': Let $\alpha \in (0, 1)$. Then

$$\sum_{n=1}^{\infty} 2^{-n\alpha} \cos 2^n x, \quad \sum_{n=1}^{\infty} 2^{-n\alpha} \sin 2^n x$$

are cts but nowhere diffentiable.

Idea: By slightly modifying the proof of Prop 4, we have for h with $|h| < \frac{c}{N}$,

$$\sigma_N(g)'(x_0 + h) = O(\log N),$$

(where we assume g is diff at x_0).

Let us prove that $\sum_{n=1}^{\infty} 2^{-n\alpha} \cos 2^n x$ is nowhere diff.

Suppose on the contrary that $F(x) := \sum_{n=1}^{\infty} 2^{-n\alpha} \cos 2^n x$ is diff at x_0.

Then for $N = 2^m$,

$$\Delta_{2N}(F)(x) - \Delta_N(F)(x) = 2^{-(m+1)\alpha} \cos(2^{m+1} x).$$

Hence

$$\Delta_{2N}(F)'(x_0+h) - \Delta_N(F)'(x_0+h)$$

$$= -2^{(m+1)(1-\alpha)} \sin(2^{m+1}(x_0+h))$$

We want to take a suitable $|h| \le \frac{c}{N}$ such that

$$2^{m+1}(x_0+h) = 2k\pi + \frac{\pi}{2}$$

for some $k \in \mathbb{Z}$

Write $\frac{2^{m+1} x_0}{2\pi} = L + t$ where $L \in \mathbb{Z}$, $t \in [0,1)$

Then letting $h_m = \frac{1}{2^{m+1}}\left[-2\pi t + \frac{\pi}{2}\right]$

Then $2^{m+1}(x_0+h_m) = 2\pi L + \frac{\pi}{2}$

We see that $\quad |h_m| \leq \dfrac{\text{const}}{N}$

Hence
$$\sigma_N'(F)(x_0+h_m) \doteq O(\log N).$$

and
$$\Delta_{2N}(F)'(x_0+h_m) - \Delta_N(F)'(x_0+h_m) = O(\log N)$$

but
$$\begin{aligned} \text{LHS} &= -\,2^{(m+1)(1-\alpha)} \sin\left(2^{m+1}(x_0+h_m)\right) \\ &= -\,2^{(m+1)(1-\alpha)} \\ &\neq O(\log N), \end{aligned}$$

leading to a contradiction. This proves Thm 3'. $\blacksquare$